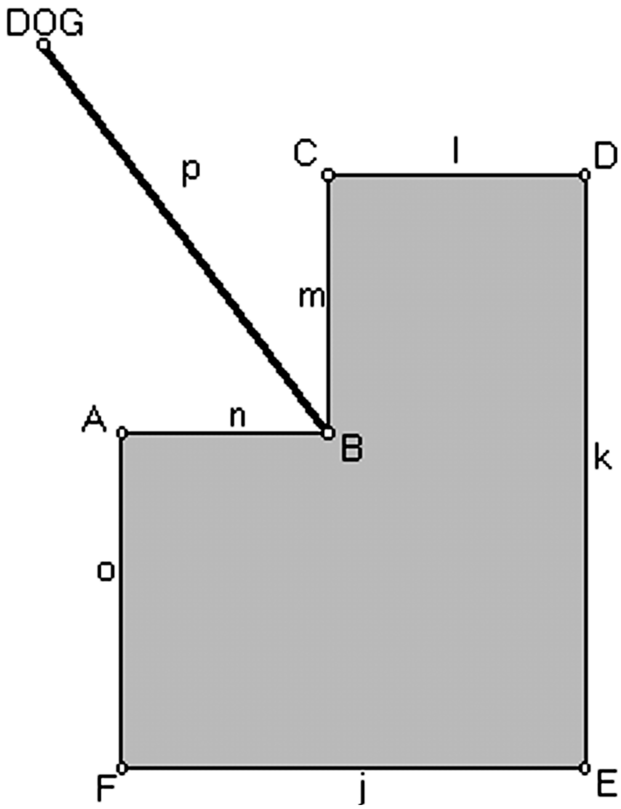
# This One's for the Dogs

Below you see a top view of a house. The segment *p* represents a leash to which a ferocious dog is attached. The owner wants to make sure the dog cannot attack visitors.




1. Figure out how much room there is on side CD and side AF of the house to build a door.
2. What is the size of the space the dog can walk around in when attached to this leash?
3. What is the length of the outside edge the dog can walk?
4. How do your answers change if the leash is attached to point C, point D and point A?
5. At what point of the building should the leash be attached so that the dog has the most/least space, the longest/shortest outer edge?
6. Change the design of the building and consider all of the above questions again. Support your answer(s) with mathematical reasons and appropriate illustrations such as drawings and/or models.

# Planning Sheet

This One's for the Dogs

**Common Core Task Alignments**

**Mathematical Practices:** MP.1, MP.4, MP.6, MP.8

**High School Content Standards:** G-C5, G-MG1, F-TF 7

### Context

This task is particularly well suited for classrooms with diverse populations because it allows for many different approaches, from actually acting the problem out in real-life scale, to making a model, to using an algebraic approach and even going as far as calculus. Many variations to the problem are possible and a few are attached.

The main concepts addressed are area and perimeter of circles and parts of circles, circle as a locus of equidistant points, instantaneous change and continuous change, rectangular and polar notation, and optimization. You may want to redesign the task in order to limit the complexity depending on your group of students, and then adding layers by revisiting the problem periodically. The use of technology is highly recommended. All drawings were done using Geometer's Sketchpad®.

### What This Task Accomplishes

With this problem, students apply knowledge of area and perimeter/circumference to a situation that can be varied in multiple ways with varying levels of complexity. This allows for many different approaches by students to expose deep and complex mathematical content and reasoning. The student should be able to generalize a given situation using correct mathematical language appropriate for her/his level of development.

### Time Required for Task

30 – 40 minutes for launching the problem and setting the context.
2 – 3 hours for the actual work.
30 – 40 minutes for summarizing the work.

It is possible to spend a full week when using some work time at home.

### Interdisciplinary Links

There are nice opportunities here to link this work with considerations for architectural and landscape designing.

## Teaching Tips

Remember that there are no numbers in the original task. These are chosen by the teacher or the students if appropriate. An alternative formulation of the task might look like: Where can you put a bowl of water so that the dog can drink?

In trying to answer some of the questions related to this task, students may do one or more of the following:

- Act out the problem in the classroom or the school building and measure area and perimeter by counting tiles (possibly) and estimating the perimeter in a similar manner.
- Make a scale model using graph paper on corrugated cardboard, and pushpins on the corners, one pushpin and string for the leash, and using estimation techniques (See attached figures).
- Make sketches of possible situations, maybe using graph paper, and apply formulas for area and perimeter of circles and wedges.
- Use integral calculus for determining area between curved paths, and length of a path.
- Generalize possible situations for this problem.

Depending on the background of your students, you may need to do some scaffolding. For an example of how to do this, see the "Possible Solutions" section.

## Suggested Materials

- Ruler and Compass
- Graph paper
- Cardboard
- Pushpins
- String
- Rope
- Scissors
- Cans
- Scale
- Geometer's Sketchpad®
- Refer to the attachments on pages 13 and 14

## Possible Solutions

What follows are some ways you might scaffold this problem with your students. Below you find two illustrations of a way to start exploring the problem on a simple rectangular "house" design. You can expand this to shapes other than quadrilaterals, such as triangles, pentagons and hexagons. Rectangular shapes are the "easiest" since they have 90-degree corners, and therefore the circle segments are always one quarter of a given circle.

**Figure 1**

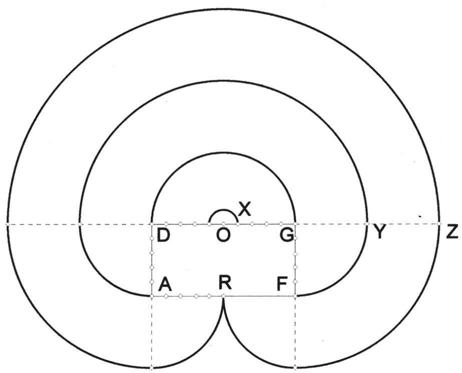
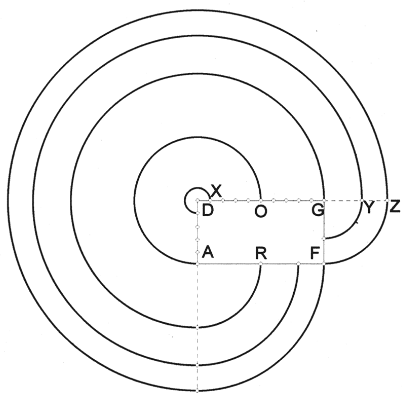



**Figure 2**

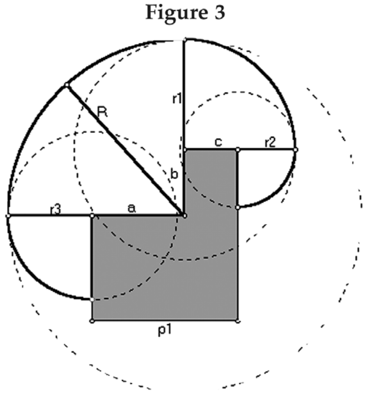



The rectangle is five by 10 units and has the leash attached to corner D in figure 1 and point O (the center of side DG) in figure 2. It would be interesting to compare area and perimeter for each leash length. In both cases, I used leash lengths of one, five, 10 and 15. The situation changes at five, 10 and 15. This is at width, length and width + length. This could be expanded as the leash gets longer. Large printouts for classroom use are included. From this investigation, students should recognize that when the leash is attached to a corner, the dog will have the most room for a given leash length.

I have included a general discussion of the problem in the task that may help you focus the classroom discussion or your coaching of groups of students. Notice that each time the leash reaches a side of the building it instantaneously becomes shorter (see figure 3). This result is a set of quarter circles when the angles in the building are 90 degrees.

In this particular design, we can distinguish the following cases:

**Figure 3**



$$R \geq a \text{ or } R < a \text{ and } R \geq b \text{ or } R < b$$

As well as:

$$R \geq b + c \text{ or } R < b + c$$

Furthermore:

$$r1 = R - b$$
$$r2 = R - (b + c)$$
$$r3 = R - a$$

There are four regions in this construction, which are all quarter circles with radius r3, R, r1, and r2 respectively.

The area and perimeter can be formulated as follows:

Area if $R > a \wedge R > (b + c)$:

$$\frac{1}{4}\pi(R^2 + (R - a)^2 + (R - b)^2 + (R - (b + c))^2)$$

For different scenarios, different terms will need to be eliminated or added.

Perimeter if $R > a \wedge R > (b + c)$:

$$\frac{1}{2}\pi(R + (R - a) + (R - b) + (R - (b + c))) = \frac{1}{2}\pi(4R - (a + 2b + c))$$

It would be very good to verify this work against actual (estimated) measurements. For example, rather than giving measurements, the students can measure the sides in centimeters and use a scale factor. The task asks for experimenting with other house designs. This is where some excellent enrichment can take place. I developed some examples with angles other than 90 degrees (See figures 4 and 5 on next page). I used Geometer's Sketchpad® software to generate these drawings. I created different designs by pulling segment BG to the right or left along line AB.

**Figure 4**

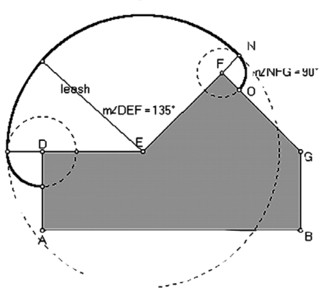
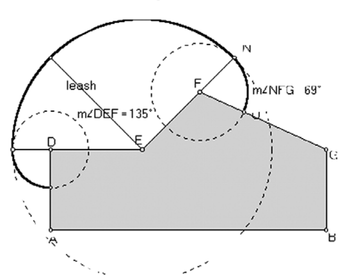



**Figure 5**



Varying the angles helps students practice calculating areas of segments with a central angle other than 90 degrees.

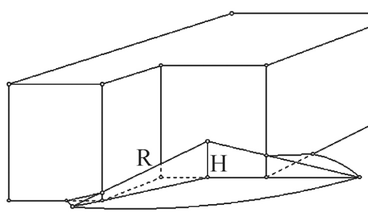
Of course, it is very helpful to change to radian measure here:

$$\text{Area} = \pi R^2 \cdot \frac{a}{2n} \text{ and Arclength} = 2\pi R \cdot \frac{a}{2n}$$

**Extensions**

3-D extension to dog and leash problem.

**Figure 6**

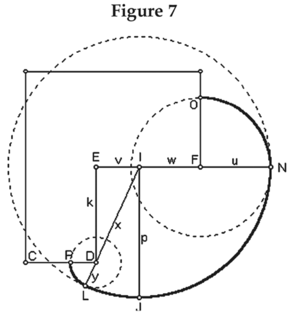



In order to calculate the area in which the dog can move, we need to consider the projection $P_R$ of the leash with length R, attached to the house at a height of H. This is done by using the Pythagorean Theorem with the following general result:

$$P_R = \sqrt{R^2 - H^2}$$

This represents the projected leash length. We can now proceed as if Pá = p, is the length of the leash on the ground. We then regard the following situation illustrated on the next page.

**Figure 7**



p is the projected length of the leash;

$$u = p - w; x = \sqrt{(v^2 + k^2)}; y = p - x$$
$$\angle \text{EID} = \angle \text{PDL} = \theta = \text{Arctan}(k / v);$$
$$\angle \text{FID} = \pi - \theta$$

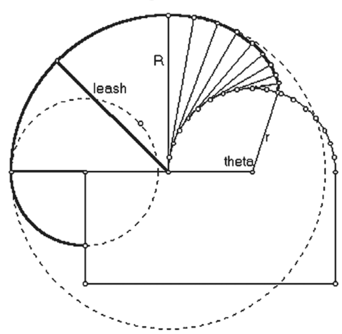
The three areas of the circle segments and triangle IED can now be found:

$$\text{Total Area} = \frac{1}{4}\pi u^2 + \frac{1}{2}[(\pi - \theta)p^2 + \theta y^3 + kv]$$

**Connecting with Calculus**

Next, try investigating house designs with a curve in them. Studying some polygon examples is useful preparation. However, in the case of the circular structure, such as a silo, the length of the leash changes continuously. You can use a tin can and a string to develop this curve on a piece of graph paper. Using Geometer's Sketchpad® to construct this design allows for approximation by polygon areas. For Calculus students, using integral calculus, surprisingly simple results appear.

**Figure 8**




This is the case of a circular edge against which the leash shortens on a continuous basis rather than at discrete moments.

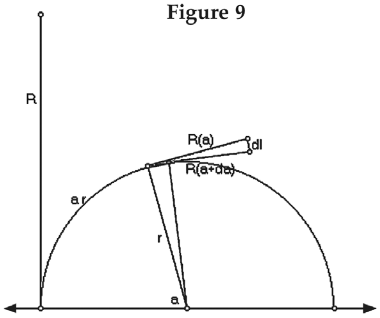
Let R be the length of the leash and r be the length of the radius of the semi-circle.

The leash will touch the circle when the arc length equals R. At this time the radius has rotated an angle of 0 radians.

The angle 0 can be calculated as follows:

$$R = 2\pi r \frac{\theta}{2\pi} = r\theta \Rightarrow \theta = \frac{R}{r} \text{ radians}$$

Figure 9




Let the length of the curve described by the shrinking radius be l and the area between the circle and this path be A. The objective is to find an expression in R and r for these two measures.

In the figure above, the angle between the two tangents is approximately equal to the angle between the two radii. When the angle is very small it is reasonable to assume that there is a wedge emanating from the center of the circle belonging to R (a), where:

$R(\alpha) = R - \alpha r$

We can calculate the length $\Delta l$ of the arc segment and the area $\Delta A$ as follows:

$\Delta l = R(\alpha_i) \bullet \Delta\alpha$ and $\Delta A = \frac{1}{2} R^2(\alpha_i) \bullet \Delta a$

We can now determine the actual length and area by using integral calculus. We integrate over angle (between 0 and R/r radians):

$$L = \int_0^{\frac{R}{r}} (R - \alpha r) d\alpha = R\alpha - \frac{1}{2}\alpha^2 r \Big|_0^{\frac{R}{r}} = \frac{R^2}{r} - \frac{R^2}{2r} = \frac{R^2}{2r} \text{ and}$$

$$A = \frac{1}{2}\int_0^{\frac{R}{r}} (R - \alpha r)^2 d\alpha = \frac{R^3}{6r}$$

I attached some printouts (Appendix A and B) from measurements done with Geometer's Sketchpad®. Theory and practice are in concordance.

The shape that has been generated by this process is called the involute of a circle. It can be expressed in polar notation as follows:

x = r cos t + rt sin t
y = r sin t + rt cos t

This is a fairly common shape used in the engineering of gears and cams. More information onthis can be found in most mechanical engineering books. One source I found was Mechanism (1939) by Keown and Faires, McGraw Hill.

## Task-Specific Assessment Notes

### Novice

The student takes some measures in centimeters. Arcs are drawn by hand, lacking accuracy, and often incorrectly. Some arcs are not extended as far as possible. The student does not seem to understand what happens at a corner. S/he does seem to have an appropriate concept of area and perimeter. The novice has a method for where the door could be built but does not take segment m into account. Overall poor communication.

### Apprentice

Although the student has an interesting approach regarding the arc length, what happens at the corners does not appear to be taken into consideration. The student reasons adequately about where to place a door and communicates clearly. S/he indicates that area was calculated using the formula for arc length. This is an incorrect procedure.

### Practitioner

The student correctly interprets the problem, particularly what happens at the corners and how this relates to quarter circle segments. S/he uses appropriate procedures and mathematical notation, with correct results. The practitioner communicates thinking clearly. His/her reasoning is supported with calculations.

### Expert

This student develops the whole problem in general, using appropriate notation and diagrams. There is one misunderstanding regarding the perimeter. Although the student makes a valid argument that the dog can cover a part of the perimeter of the house that is always twice the length of the leash, it does not answer the question in the task.

# Appendix A

AG = 5.20 inches

DA = 2.60 inches

m∠BDA = 2.01 radians

Area AGA'' = 4.23 inches$^2$

Area A'A''A''' = 2.84 inches$^2$

Area A''A'''A'''' = 1.72 inches$^2$

Area A'''A''''A''''' = 0.88 inches$^2$

Area A''''A'''''A'''''' = 0.31 inches$^2$

sum = 9.97 square inches

theoretical value $\frac{EG^3}{(6 \cdot DA)}$ = 9.01 inches$^2$

Arc length $\overset{\frown}{c2A''G}$ = 1.66 inches

Arc length $\overset{\frown}{c3A'''A''}$ = 1.34 inches

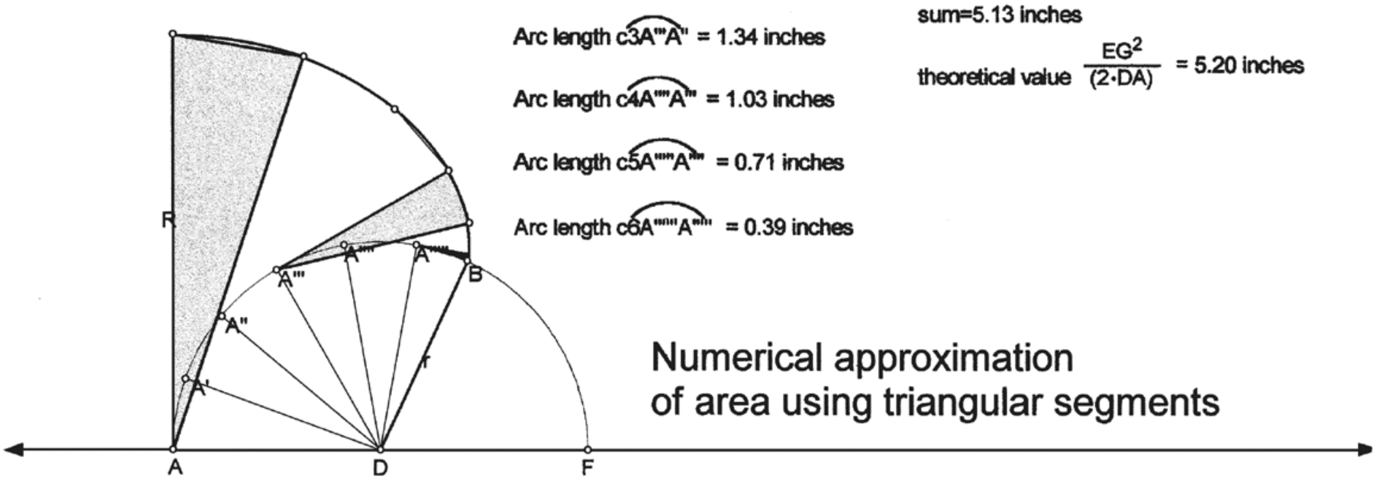
Arc length $\overset{\frown}{c4A''''A'''}$ = 1.03 inches

Arc length $\overset{\frown}{c5A'''''A''''}$ = 0.71 inches

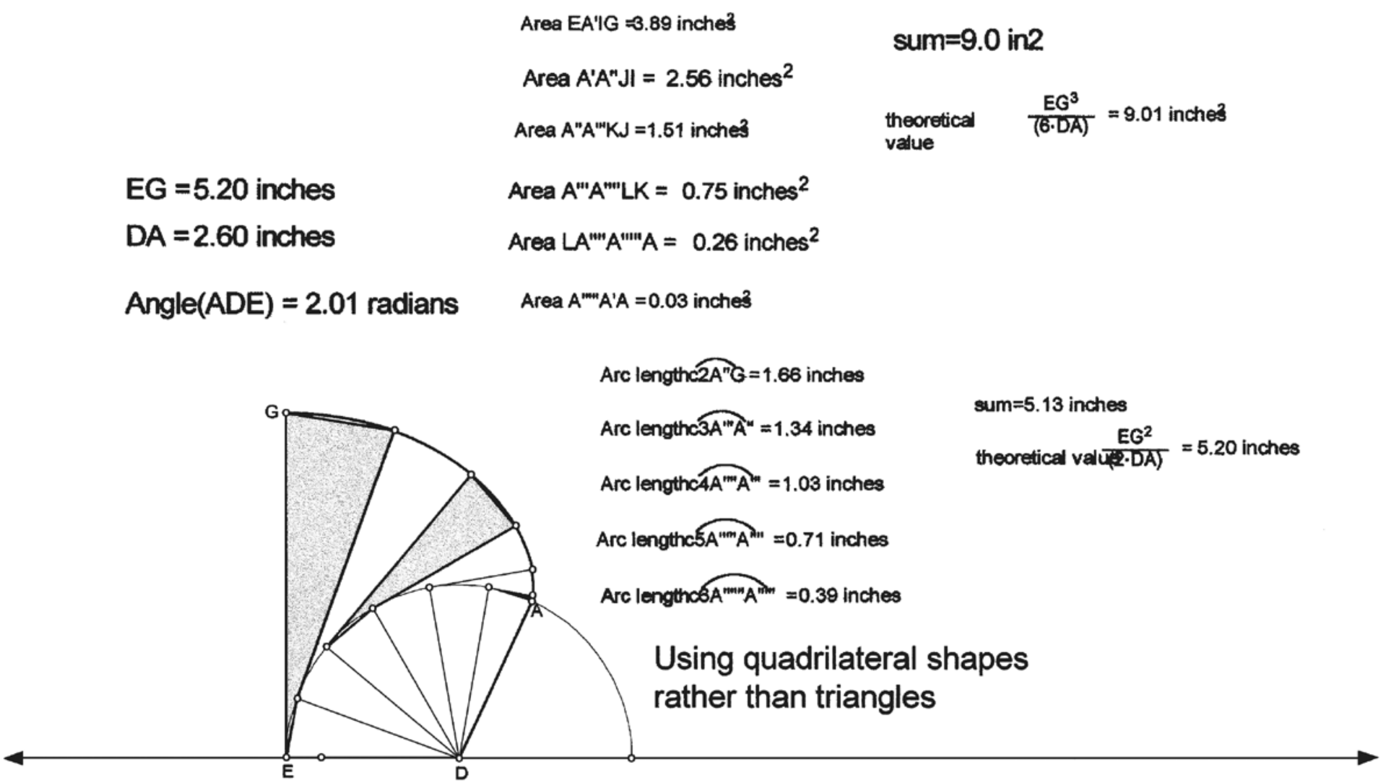
Arc length $\overset{\frown}{c6A''''''A'''''}$ = 0.39 inches

sum=5.13 inches

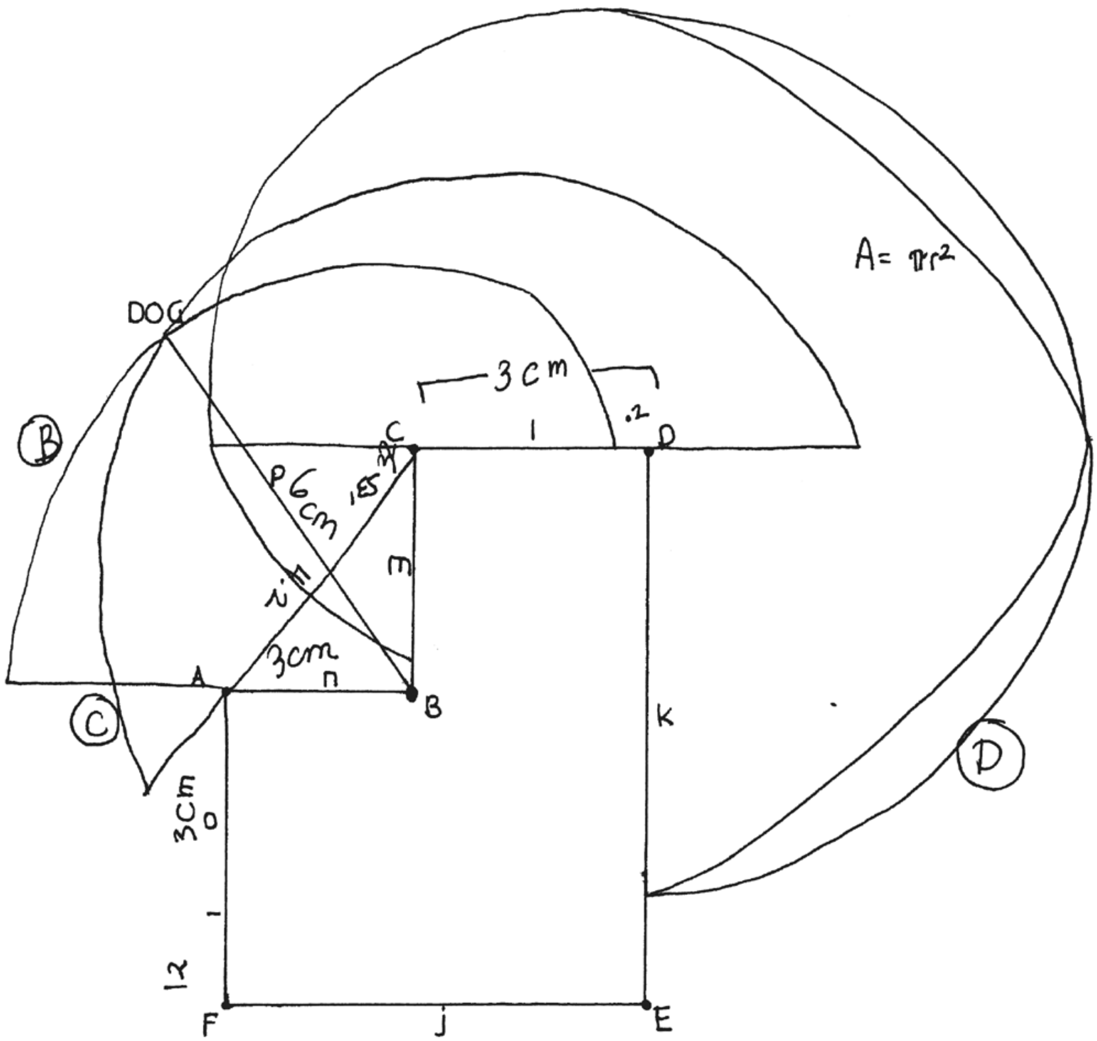
theoretical value $\frac{EG^2}{(2 \cdot DA)}$ = 5.20 inches



# Appendix B


Area EA'IG = 3.89 inches²
Area A'A"JI = 2.56 inches²
Area A"A"'KJ = 1.51 inches²
Area A"'A""LK = 0.75 inches²
Area LA""A'""A = 0.26 inches²
Area A'""A'A = 0.03 inches²
sum=9.0 in2
theoretical value EG³/(6·DA) = 9.01 inches³
EG = 5.20 inches
DA = 2.60 inches
Angle(ADE) = 2.01 radians
Arc length c2 A"G = 1.66 inches
Arc length c3 A"'A" = 1.34 inches
Arc length c4 A""A"' = 1.03 inches
Arc length c5 A'""A"" = 0.71 inches
Arc length c6 A""""A'"" = 0.39 inches
sum=5.13 inches
theoretical value EG²/(2·DA) = 5.20 inches
G
A
E
D
Using quadrilateral shapes rather than triangles

## Novice



## Novice

1. On Side CD you have .2 Cm space left
   On Side FA you have 1.2 cm space left
   You have more room on side FA.
2. 66 = 36 Cm

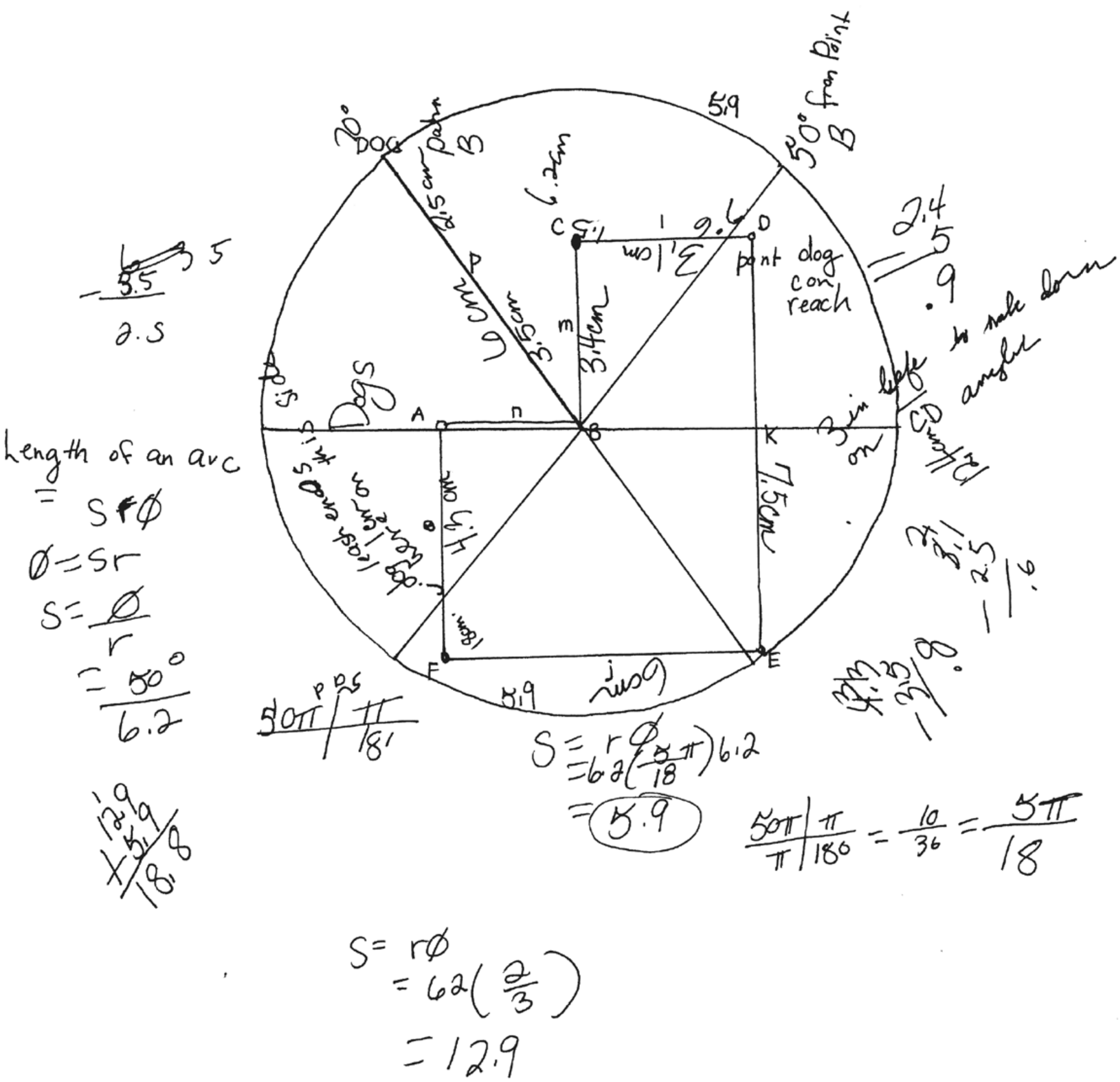
3. On CD thedog can walk 2.8 cm
   On BC the dog can walk 3.3 cm
   On AB thedog can walk 3 cm
   On AF The dog can walk 3 cm

4. If leashed to pt C:
   You have 2.5 cm left on side AF to build a door

The student uses unrealistic measurements.

All results are inferred from an inaccurate or incomplete diagram.

## Apprentice



5.9
50° from Point B
70°
DOG
3.5 cm from B
6.2 cm
C
D
3.1 cm
point dog can reach
2.4
5
.9
3 in left on to make down angle
3.4 cm
3.5 cm
6 cm
P
m
Dogs
A
n
B
K
7.5 cm
Length of an arc
= S r Ø
Ø = Sr
S = Ø / r
= 50° / 6.2
4.1 cm
F
E
5.9
6 cm
50π / π/18
S = rØ
= 6.2(5/18 π) 6.2
= 5.9
50π/π / 180 = 10/36 = 5π/18
12.9
x 5.9
18.8
S = rØ
= 62(2/3)
= 12.9

Exemplars® K–12
We Set the Standards!

## Apprentice

Student communicates clearly and appropriately.

Incorrect results based on inappropriate procedure.

Does not take into account what happens at corners of the building.

This One's for the Dogs

First, I began by measuring all sides, including the bisecting arc segment, p. $\overline{AB}$ = 2.5cm, $\overline{AF}$ = 4.3cm, $\overline{CB}$ = 3.4cm, $\overline{CD}$ = 3.1cm, $\overline{DE}$ = 7.5cm, $\overline{FE}$ = 6cm., $\overline{SP}$ = 6cm. From the point where the p is placed on segment $\overline{p}$, there was a measurement of 3.5 cm, leaving another 2.5 cm. from the placed p to the actual dog. The 3.5 cm piece of $\overline{p}$ generally fit to $\overline{CB}$ with an overlay of .1cm and to $\overline{AB}$ with an overlay of 1cm. All of this was done in order to find out how far the dog could go. Any left over space was then put into ~~the~~ the segments $\overline{CD}$ or $\overline{AF}$. Then, it was determined that on $\overline{CD}$, there ~~was~~ were only .6cm space left for a door, so it would probably not be wise to put a door there, Considering the added unknown, the size of the dog. On segment $\overline{AF}$ there were .8 cm left over, so also not a good decision to put a door. It would perhaps be best to attach the dog to point E, is considering putting a door on $\overline{CD}$ or $\overline{AF}$. The area that was available to the dog was 18.8 cm², determined using the area of an arc S=rØ. One arc was 50° and another came to 120°. Room must then be allowed for error, since the supplementary angles should equal 180°, yet only equal 170°. Anyways, the total arc areas of 50° and 120° were added together to find the dog's movable space.

Exemplars® K–12
We Set the Standards!

## Practitioner

1. I subtrated the length of the leash from the length of the wall, after that we subtracted the remaining length of leash from the next wall. to find how much room would be left for a door. CD = 0.5 cm left AF = 0.9 cm

2. Since I knew the radius of each, I divided up the ~~house~~ in Area into 3 1/4 circles, & found the Area in each, & found the total by adding them together.

$A = \pi r^2$

$A_1 = \frac{\pi\ 2.7^2}{4} = 1.823\pi$

$A_2 = \frac{\pi\ 6^2}{4} = 9\pi$

$A_3 = \frac{\pi\ 3.4^2}{4} = 2.89\pi$

$AT = 43.08\text{cm}^3$

Clear communication and mathematical notation.

Correct procedures and results.

Demonstrates understanding and mathematical reasoning.

3. I took the 3 same 1/4 circles, & found the circumference of each.

$C = 2\pi r$

$C = \frac{1}{2}\pi(2.7) + \frac{1}{2}\pi 6 + \frac{1}{2}\pi 3.4 = 19.007$

4. at all points, other than B would increase the area the dog would be allowed to roam, due to the increase of radius

5. The least point will B, because the walls decrease the radius, the ~~walls~~ walls restrict the area. D would

## Practitioner

give the dog the most room, because it has the most area, because it has 3/4 of a circle, plus the area in the remaining area.

$A = \pi r^2$

* off pt. D $A = \frac{3}{4}\pi 6^2 = 84.823\text{cm}^3 + A_2 = \frac{1}{4}\pi 2.8^2 = 6.158$

$A_T = 90.98\text{ cm}^3$

pt. E $A_1 = \frac{3}{4}\pi 6^2 = 84.823\text{cm} + A = \frac{1}{4}\pi .2^2 = .0314\text{cm}^3$

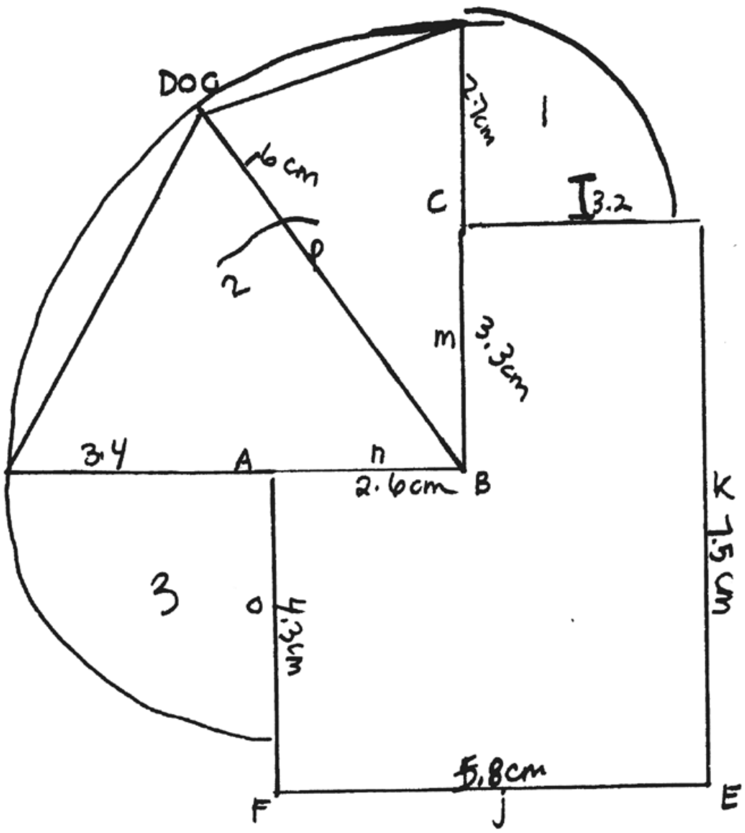
$A_T = 84.854$

pt. F $A_1 = \frac{3}{4}\pi 6^2 = 84.823 + A_2 = \frac{1}{4}\pi .2^2 = .0314$

$+ A_3 = \frac{1}{4}\pi 1.7^2 = 2.2698 \quad A_T = 87.123\text{cm}^3$

Pt. A B C would not, because they are also constructed by the walls, they do not even make an full 3/4 circle w/ a radius of 6.

## Practitioner




$A = \pi r^2$

$A_1 = \dfrac{\pi 2.7^2}{4} = 1.823\pi$

$A_2 = \dfrac{\pi 6^2}{4} = 9\pi$

$A_3 = \dfrac{\pi 3.4^2}{4} = 2.89\pi$

$43.08$

$C = 2\pi r$

$C_2 = \dfrac{1\pi 2.7}{2} + \frac{1}{2}\pi 6 + \frac{1}{2}\pi 3.4 = 19.007$

Exemplars® K–12
We Set the Standards!

## Expert

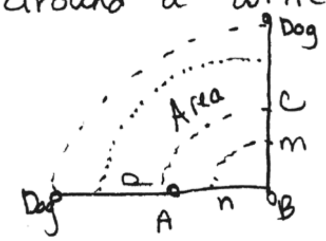
① When the leash swings over to side CB, the amount of overlap leash is P-M. Therefore, on side 1, the room we have to build a door is

$$r = L - (p-m)$$

Similarly, when the leash swings over to side AB, the overlap leash length is p-n. Therefore, the room on side AF that we have to build a door is

$$r = O - (p-n)$$

② First, we must consider the area the dog can move in when the leash does not curve around a corner



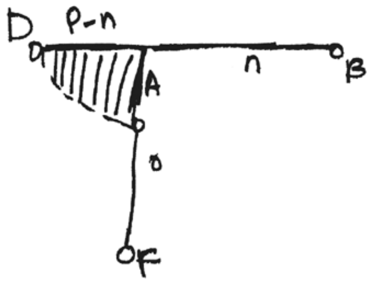

This is one quarter the area of a circle with a radius of p. Therefore; the area of this section is

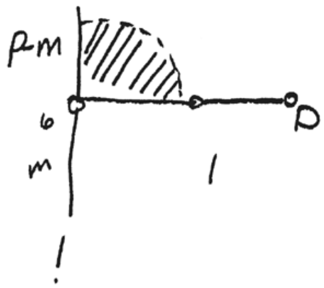
$$A = \frac{1}{4}(\pi p^2)$$

Next, we must consider the area of the two quarter circles that the overlap leash lengths can create around the corners.

Student discusses the problem in general using clear and appropriate symbolic representation.

## Expert




The areas of these quarter circles are

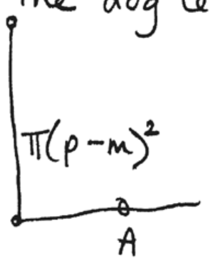
$$A = \frac{\pi}{4}(p-n)^2$$

and

$$A = \frac{\pi}{4}(p-M)^2$$

There fore, the total area the dog leash walk about is

$$\frac{+\pi(p-n) \quad \pi(p-m)^2}{4}$$




(3) All we must do is add together the inner Sides CB and AB and then the amounts of sides CD and AF that the dog can run

$$\text{Length} = m + n + (p-m) + (p-n)$$

$$\text{Length} = 2p$$

## Expert

the leash is attached to points C and A, where the areas would still be that of a semi circle with radius P and even more besides.
The outer edge however, never changes anywhere the leash is attached. At point A, point B, point C and point D, there is nothing to prevent the dog from extending the leash to its full extent on either side. Therefore, the length of outer edge the dog can walk will always equal twice the length of the leash.

length = 2P

# Exemplars® Classic 5-Criteria Math Rubric*

| | Problem Solving | Reasoning and Proof | Communication | Connections | Representation |
|---|---|---|---|---|---|
| **Novice** | No strategy is chosen, or a strategy is chosen that will not lead to a solution.<br><br>Little or no evidence of engagement in the task present. | Arguments are made with no mathematical basis.<br><br>No correct reasoning nor justification for reasoning is present. | No awareness of audience or purpose is communicated.<br>or<br>Little or no communication of an approach is evident<br>or<br>Everyday, familiar language is used to communicate ideas. | No connections are made. | No attempt is made to construct a mathematical representation. |
| **Apprentice** | A partially correct strategy is chosen, or a correct strategy for only solving part of the task is chosen.<br><br>Evidence of drawing on some previous knowledge is present, showing some relevant engagement in the task. | Arguments are made with some mathematical basis.<br><br>Some correct reasoning or justification for reasoning is present with trial and error, or unsystematic trying of several cases. | Some awareness of audience or purpose is communicated, and may take place in the form of paraphrasing of the task.<br>or<br>Some communication of an approach is evident through verbal/written accounts and explanations, use of diagrams or objects, writing, and using mathematical symbols.<br>or<br>Some formal math language is used, and examples are provided to communicate ideas.<br><br>NOTE: *The following are not assessed:*<br>• *Numbers and their names (i.e., 5, five, etc.)*<br>• *Verbs (i.e., counted, divided, etc.)*<br>• *Generic symbols (+, –, ×, ÷, =)* | Some attempt to relate the task to other subjects or to own interests and experiences is made. | An attempt is made to construct a mathematical representation to record and communicate problem solving. |

# Exemplars® Classic 5-Criteria Math Rubric (Cont.)*

| | Problem Solving | Reasoning and Proof | Communication | Connections | Representation |
|---|---|---|---|---|---|
| **Practitioner** | A correct strategy is chosen based on mathematical situation in the task.<br><br>Planning or monitoring of strategy is evident.<br><br>Evidence of solidifying prior knowledge and applying it to the problem solving situation is present.<br><br>*Note: The Practitioner must achieve a correct answer.* | Arguments are constructed with adequate mathematical basis.<br><br>A systematic approach and/or justification of correct reasoning is present. This will lead to connections. | A sense of audience or purpose is communicated.<br>and/or<br>Communication of an approach is evident through a methodical, organized, coherent sequenced and labeled response.<br><br>Formal math language is used throughout the solution to share and clarify ideas.<br><br>NOTE: *The following are not assessed:*<br>• *Numbers and their names (i.e., 5, five, etc.)*<br>• *Verbs (i.e., counted, divided, etc.)*<br>• *Generic symbols (+, –, ×, ÷, =)* | Mathematical connections or observations are recognized. Some examples may include, but are not limited to:<br>• clarification of the task.<br>• exploration of mathematical phenomenon.<br>• noting patterns, structures and regularities. | An appropriate and accurate mathematical representation(s) is constructed and refined to solve problems or portray solutions. |
| **Expert** | An efficient strategy is chosen and progress towards a solution is evaluated.<br><br>Adjustments in strategy, if necessary, are made along the way, and/or alternative strategies are considered.<br><br>Evidence of analyzing the situation in mathematical terms, and extending prior knowledge is present.<br><br>*Note: The Expert must achieve a correct answer.* | Deductive arguments are used to justify decisions and may result in formal proofs.<br><br>Evidence is used to justify and support decisions made and conclusions reached. This will lead to connections. | A sense of audience and purpose is communicated.<br>and/or<br>Communication at the Practitioner level is achieved, and communication of argument is supported by mathematical properties.<br><br>Precise math language and symbolic notation are used to consolidate math thinking and to communicate ideas.<br><br>NOTE: *The following are not assessed:*<br>• *Numbers and their names (i.e., 5, five, etc.)*<br>• *Verbs (i.e., counted, divided, etc.)*<br>• *Generic symbols (+, –, ×, ÷, =)* | Mathematical connections or observations are used to extend the solution to other mathematics or to a deeper understanding of mathematics. Some examples may include, but are not limited to:<br>• testing and accepting or rejecting of a hypothesis or conjecture.<br>• explanation of phenomenon.<br>• generalizing and extending the solution to other cases. | An abstract or symbolic mathematical representation(s) is constructed to analyze relationships, and to clarify or interpret phenomenon. |